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Integrability and algebraic limit cycles for polynomial differential systems with homogeneous nonlinearities ☆

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Dedicated to Professor George Sell's on the occasion of his 65th birthday

Abstract

We consider the class of polynomial differential equations $\dot{x} = \lambda x - y + P_n(x, y)$, $\dot{y} = x + \lambda y + Q_n(x, y)$, where P_n and Q_n are homogeneous polynomials of degree n. These systems have a focus at the origin if $\lambda \neq 0$, and have either a center or a focus if $\lambda = 0$. Inside this class we identify a new subclass of Darbouxian integrable systems having either a focus or a center at the origin. Additionally, under generic conditions such Darbouxian integrable systems can have at most one limit cycle, and when it exists is algebraic. For the case $n = 2$ and 3, we present new classes of Darbouxian integrable systems having a focus.

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1. Introduction and statement of the results

Three of the main problems in the qualitative theory of real planar differential systems are the determination of centers, limit cycles and first integrals. This paper deals mainly with the determination of first integrals and limit cycles.

As usual a *center* is a singular point having a neighborhood filled of periodic orbits and a *focus* is a singular point having a neighborhood where all the orbits spiral in forward or in backward time to it.

In this paper we study the class of real planar polynomial differential systems of the form

$$\dot{x} = \lambda x - y + P_n(x,y), \quad \dot{y} = x + \lambda y + Q_n(x,y), \tag{1}$$

where P_n and Q_n are homogeneous polynomials of degree n. Inside this class we will characterize a new subclass having either a focus or a center at the origin which is Darbouxian integrable.

In order to be more precise we need some preliminary notation and results. Thus, in polar coordinates (r,θ), defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{2}$$

system (1) becomes

$$\dot{r} = \lambda r + f(\theta)r^n, \quad \dot{\theta} = 1 + g(\theta)r^{n-1}, \tag{3}$$

where

$$f(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$

$$g(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta).$$

We remark that f and g are homogeneous trigonometric polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region

$$R = \{(r,\theta):\ 1 + g(\theta)r^{n-1} > 0\}$$

the differential system (3) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\lambda r + f(\theta)r^n}{1 + g(\theta)r^{n-1}}. \tag{4}$$

It is known that the periodic orbits surrounding the origin of system (3) do not intersect the curve $\dot{\theta} = 0$ (see, for instance, the Appendix of [2]). Therefore, these periodic orbits are contained in the region R, and consequently they also are periodic orbits of the differential equation (4).

The transformation $(r,\theta)\mapsto(\rho,\theta)$ defined by

$$\rho = \frac{r^{n-1}}{1+g(\theta)r^{n-1}} \tag{5}$$

is a diffeomorphism from the region R into its image. As far as we know the first to use this transformation was Cherkas in [4]. If we write Eq. (4) in the variable ρ, we obtain the following Abel differential equation:

$$\begin{aligned}\frac{d\rho}{d\theta} &= (n-1)g(\theta)[\lambda g(\theta)-f(\theta)]\rho^3 + [(n-1)(f(\theta)-2\lambda g(\theta)) - g'(\theta)]\rho^2 + (n-1)\lambda\rho \\ &= A(\theta)\rho^3 + B(\theta)\rho^2 + C\rho. \end{aligned} \tag{6}$$

These kinds of differential equations appeared in the studies of Abel on the theory of elliptic functions. For more details on Abel differential equations, see [5,9,11].

Here we shall consider the Abel differential equation (6) defined on the cylinder $(\rho,\theta)\in\mathbb{R}\times\mathbf{S}^1$, where as usual $\mathbb{R}$ denotes the set of real numbers, and $\mathbf{S}^1$ denotes the circle. Of course, only the orbits of the half-cylinder $\rho>0$ can come from the orbits of system (1). Note that the origin of system (1) plays the role of the periodic orbit $\rho=0$ for the Abel differential equation (6).

In short, from [2] we have the following result.

Proposition 1. *The function $r=r(\theta)$ is a periodic solution of system* (3) *surrounding the origin if and only if $\rho(\theta)=r(\theta)^{n-1}/(1+g(\theta)r(\theta)^{n-1})$ is a periodic solution of the Abel differential equation* (6).

We shall say that all systems (1) define the class HN, we use the letters HN for homogeneous nonlinearities. A system of the class HN belongs to the subclass HN_f if and only if its functions $A(\theta)$ and $B(\theta)$ (defined in (6)) are such that the following equality holds:

$$A'(\theta)B(\theta) - A(\theta)B'(\theta) = aB(\theta)^3 - A(\theta)B(\theta)C, \tag{7}$$

for some $a\in\mathbb{R}$.

We shall prove that all systems (1) of the subclass HN_f have a focus or center at the origin with a Darbouxian first integral.

In [14] it was studied the subclass HN_* of HN. A system of the class HN belongs to the subclass HN_* if and only if $\lambda=0$ and the functions $A(\theta)$ and $B(\theta)$ are such that

(i) either $A(\theta)$ changes of sign, or $f(\theta)\equiv 0$, or $g(\theta)\equiv 0$ and $\int_0^{2\pi} f(\theta)\,d\theta = 0$;
(ii) either $B(\theta)/\gcd\{A(\theta),B(\theta)\}$ does not vanish, or $B(\theta)\equiv 0$; and
(iii) for some $a\in\mathbb{R}$ the equality $A'(\theta)B(\theta)-A(\theta)B'(\theta)=aB(\theta)^3$ holds.

In [14] it was also proved that all systems (1) of the subclass HN_* have a center at the origin. Since $C=(n-1)\lambda$, it is easy to see that all systems (1) of the subclass

HN_* belongs to the subclass HN_f. In a previous paper [13], the authors of [14] considered three subfamilies in HN_* and it was also studied the limit cycles that bifurcate from the periodic orbits of these centers.

We must mention that we have found the subclass HN_f thanks to cases (a), (b), (c) and (d) of Abel differential equations studied in Kamke [11, pp. 24,25].

A function of the form $f_1^{\lambda_1}\dots f_p^{\lambda_p}\exp(h/g)$, where f_i, g and h are polynomials in $\mathbf{C}[x,y]$ and the λ_i's are complex numbers, is called a *Darbouxian function*. System (1) is called *Darbouxian integrable* if the system has a first integral or an integrating factor which is a Darbouxian function (for a definition of a first integral and of an integrating factor, see for instance [3,7]). Our three main results are the following two theorems and two corollaries.

Theorem 2. *The following statements hold.*

(a) *Systems* (1) *in the class HN_f with $\lambda\neq 0$ and $A(\theta)B(\theta)\neq 0$ have a focus at the origin with the Darbouxian first integral $\tilde{H}(x,y)$ obtained from*

$$H(\rho,\theta)=\begin{cases}\dfrac{\rho\,\exp((1-n)\lambda\theta)\,\exp[-\frac{1}{\sqrt{4a-1}}\ \arctan[\frac{(1+2\rho A(\theta)/B(\theta))}{\sqrt{4a-1}}]]}{\sqrt{\rho^2A(\theta)^2/B(\theta)^2+\rho A(\theta)/B(\theta)+a}} & \text{if } a>\frac{1}{4},\\[2ex] \dfrac{\rho\,\exp((1-n)\lambda\theta)\,\exp(\frac{1}{1+2\rho A(\theta)/B(\theta)})}{1+2\rho A(\theta)/B(\theta)} & \text{if } a=\frac{1}{4},\\[2ex] \dfrac{\rho\,\exp((1-n)\lambda\theta)|\sqrt{1-4a}+1+2\rho A(\theta)/B(\theta)|^{\frac{1}{2}(-1+\frac{1}{\sqrt{1-4a}})}}{|\sqrt{1-4a}-1-2\rho A(\theta)/B(\theta)|^{\frac{1}{2}(1+\frac{1}{\sqrt{1-4a}})}} & \text{if } a<\frac{1}{4},\ a\neq 0,\\[2ex] \dfrac{\exp((1-n)\lambda\theta)B(\theta)}{A(\theta)} & \text{if } a=0,\end{cases}$$

through the changes of variables (2) *and* (5).

(b) *Systems* (1) *in the class HN_f with $\lambda\neq 0$ and $a=A(\theta)B(\theta)=0$ have a focus at the origin with the first integral $\tilde{H}(x,y)$ obtained from*

$$H(\rho,\theta)=\begin{cases}\frac{1}{\rho}\big[\exp((n-1)\lambda\theta)+\rho\int\exp((n-1)\lambda\theta)B(\theta)\,d\theta\big] & \text{if } A(\theta)=0,\\ \frac{1}{\rho^2}\big[\exp(2(n-1)\lambda\theta)+2\rho^2\int\exp(2(n-1)\lambda\theta)A(\theta)\,d\theta\big] & \text{if } B(\theta)=0,\end{cases}$$

through the changes of variables (2) *and* (5). *Moreover, these systems are Darbouxian integrable.*

(c) *Systems* (1) *in the class HN_f with $\lambda=0$ have a center at the origin and the analytic first integral $\tilde{H}(x,y)$ can be obtained from the expressions of $H(\rho,\theta)$ in statements* (a) *and* (b) *taking $\lambda=0$.*

(d) *The following systems in the class HN_f with $\lambda=0$ have a center at the origin and a rational first integral defined in its neighborhood*:

(d.1) *The systems with $f(\theta)\equiv 0$ which implies $A(\theta)=0$.*

(d.2) *The systems with $g(\theta)\equiv 0$ which implies $A(\theta)=0$ and $\int_0^{2\pi}f(\theta)\,d\theta=0$.*

(d.3) *The systems with $B(\theta)\equiv 0$.*

(d.4) *The systems whose* a (*defined in* (7)) *satisfies* $a<1/4$, $a\neq 0$ *and* $\sqrt{1-4a}$ *is rational.*

Theorem 2 will be proved in Section 2. Statement (c) appears in [14], here it follows easily from statements (a) and (b), but it was not known in [14] that these systems were Darbouxian integrable.

Systems (1) with $n=2$ are *quadratic systems* having a focus or a center at the origin. Quadratic systems have been studied intensively, there are more than 1000 papers on them, see for instance [17]. In the first corollary of the appendix we compute the class of quadratic systems satisfying statements (a) and (b) of Theorem 2; i.e. we provide new classes of Darbouxian integrable quadratic systems this time with a focus. As far as we know only one class of Darbouxian integrable quadratic systems with simultaneously a focus and a center was known until now, see [15,20].

System (1) with $n=3$ is a linear center perturbed by cubic homogeneous polynomials. Following the work of Sibirskii [18], this system with a focus or a center and with cubic homogeneous nonlinearities may be written through an affine change of variables and a time rescaling into the form

$$\begin{aligned}\dot{x} &= \lambda_1 x - y - \lambda_2 x^3 + \lambda_3 x^2 y + \lambda_4 x y^2 + \lambda_5 y^3,\\ \dot{y} &= x + \lambda_1 y + \lambda_6 x^3 + \lambda_7 x^2 y - \lambda_3 x y^2 + \lambda_8 y^3,\end{aligned} \tag{8}$$

where λ_i are arbitrary constants for $i=1,\dots,8$. In the second corollary of the appendix we provide new classes of Darbouxian integrable systems (8) having either a focus or a center at the origin.

A *limit cycle* of system (1) is a periodic orbit of this system isolated in the set its periodic orbits, see [21] for more details. We say that a limit cycle γ is *algebraic* if it is contained in an algebraic curve.

Theorem 3. *If systems* (1) *in the class* HN_f *with* $\lambda\neq 0$ *and* $A(\theta)B(\theta)\neq 0$ *have limit cycles, then they are algebraic. Moreover, such systems can have at most one limit cycle, and there are systems with* 0 *or* 1 *limit cycle.*

We remark that in the proof of Theorem 3 in Section 3 we will provide the explicit expression of the algebraic limit cycle.

2. Proof of Theorem 2

In this section we prove the four statements of Theorem 2.

Proof of Theorem 2(a). Following the case (d) of Abel differential equation studied in [11, p. 25], we do the change of variables $(\rho,\theta)\to(\eta,\xi)$ defined by $\rho=u(\theta)\eta(\xi)$, where $u(\theta)=\exp(\int C\,d\theta)$ and $\xi=\int u(\theta)B(\theta)\,d\theta$. This transformation writes the

Abel differential equation (6) into the form

$$\eta'(\xi) = g(\xi)[\eta(\xi)]^3 + [\eta(\xi)]^2, \tag{9}$$

where $g(\xi) = u(\theta)A(\theta)/B(\theta)$ and $' = d/d\xi$. As $C = (n-1)\lambda$ we have that $u(\theta) = \exp((n-1)\lambda\theta)$ and therefore $\xi = \int \exp((n-1)\lambda\theta)B(\theta)\,d\theta$ and $g(\xi) = \exp((n-1)\lambda\theta)A(\theta)/B(\theta)$.

Doing the change $\xi \to t$ in the independent variable defined by $\xi' = -1/(t\eta(\xi))$, where now $' = d/dt$, Eq. (9) takes the form

$$t^2\xi''(t) + g(\xi(t)) = 0. \tag{10}$$

This nonlinear ordinary differential equation, in the particular case that $g(\xi) = a\xi$, is an Euler differential equation. Then, doing the change $t \to \tau$ of the independent variable given by $t = \exp(\tau)$, Eq. (10) becomes the linear ordinary differential equation with constant coefficients

$$\xi''(\tau) - \xi'(\tau) + a\xi(\tau) = 0, \tag{11}$$

where here $' = d/d\tau$. Eq. (11) has the characteristic equation $k^2 - k + a = 0$, therefore its general solution is $\xi(\tau) = C_1\exp(\tau/2) + C_2\tau\exp(\tau/2)$ if $a = 1/4$, and $\xi(\tau) = C_1\exp(k_1\tau) + C_2\exp(k_2\tau)$ if $a \neq 1/4$, where k_1 and k_2 are the roots of the characteristic equation. Going back to the independent variable $t = \exp(\tau)$ the solution of the Euler differential equation is $\xi(t) = C_1\sqrt{t} + C_2\sqrt{t}\ln t$ if $a = 1/4$, and $\xi(t) = C_1t^{k_1} + C_2t^{k_2}$ if $a \neq 1/4$. Note that $g(\xi) = a\xi$ means $\exp((n-1)\lambda\theta)A(\theta)/B(\theta) = a\int \exp((n-1)\lambda\theta)B(\theta)\,d\theta$, or equivalently derivating respect to θ we get

$$\frac{d}{d\theta}\left(\frac{A(\theta)}{B(\theta)}\right) = aB(\theta) - \frac{A(\theta)C}{B(\theta)}, \tag{12}$$

which is equivalent to condition (7). Finally, going back through the change of variables to the variables (ρ, θ) and taking into account if the roots k_1 and k_2 are real or complex, we obtain the first integrals of statement (a) according to the values of a.

Now, we are going to prove that systems of statement (a) are Darbouxian integrable. To see this we are going to prove that all the terms that appear in the first integral of those systems are of the form $f_1^{\lambda_1}\dots f_p^{\lambda_p}$ with f_i a polynomial and λ_i a complex number. First, the term $\exp((1-n)\lambda\theta)$, takes the form

$$\exp((1-n)\lambda\theta) = \exp((1-n)\lambda\ \arctan(y/x)) = (x+iy)^{i(n-1)\lambda/2}(x-iy)^{-i(n-1)\lambda/2}.$$

We recall that for $f \in \mathbf{C}[x,y]$, if $f = 0$ is an invariant algebraic curve of a real polynomial differential system, then its complex conjugate $\bar{f} = 0$ is also an invariant algebraic curve, see for instance [7]. Therefore, if among the invariant algebraic curves of system (1) a complex conjugate pair $f=0$ and $\bar{f}=0$ occurs, then the first integral has a factor of the form $f^{\mu}\bar{f}^{\bar{\mu}}$, which is the (multi-valued) real function

$$[(\mathrm{Re} f)^2+(\mathrm{Im} f)^2]^{\mathrm{Re}\mu}\exp\left(-2\,\mathrm{Im}\mu\ \arctan\left(\frac{\mathrm{Im} f}{\mathrm{Re} f}\right)\right).$$

On the other hand, writing $\rho=r^{n+1}/(r^2+g(\theta)r^{n+1})$, it follows that

$$F=\frac{B(\theta)+2\rho A(\theta)}{B(\theta)}=\frac{r^{n+1}[(r^2+g(\theta)r^{n+1})B(\theta)+2r^{n+1}A(\theta)]}{(r^2+g(\theta)r^{n+1})r^{n+1}B(\theta)}$$

is a rational function in cartesian coordinates because $g(\theta)$, $A(\theta)$ and $B(\theta)$ are homogeneous trigonometric polynomials of degree $n+1$, $2(n+1)$ and $n+1$, respectively. Hence, taking into account these relations the first integral for $a>1/4$ is given by the Darbouxian function

$$H(\rho,\theta)=\rho\exp((1-n)\lambda\theta)f^{\mu}\bar{f}^{\bar{\mu}},$$

where $f=\mathrm{Re} f+i\,\mathrm{Im} f=F+i\sqrt{4a-1}$ and $\mu=\mathrm{Re}\mu+i\,\mathrm{Im}\mu=-1/2+i\,1/(2\sqrt{4a-1})$. The first integral for $a=1/4$ is the Darbouxian function

$$H(\rho,\theta)=\rho\exp((1-n)\lambda\theta)\exp(1/F)/F.$$

The first integral for $a<1/4$ and $a\neq 0$ is the Darbouxian function

$$H(\rho,\theta)=\rho\exp((1-n)\lambda\theta)|\sqrt{1-4a}+F|^{\mu_1}|\sqrt{1-4a}-F|^{\mu_2},$$

where $\mu_1=(-1+1/\sqrt{1-4a})/2$ and $\mu_2=(1+1/\sqrt{1-4a})/2$. Finally, the first integral for $a=0$ is the Darbouxian function

$$H(\rho,\theta)=\frac{\exp((1-n)\lambda\theta)A(\theta)r^{2(n+1)}}{r^{n+1}B(\theta)r^{n+1}},$$

and this completes the proof of statement (a). □

Proof of Theorem 2(b). If $A(\theta)\equiv 0$ or $B(\theta)\equiv 0$, the Abel differential equation (6) is the Bernoulli differential equation $d\rho/d\theta=B(\theta)\rho^2+C\rho$, or $d\rho/d\theta=A(\theta)\rho^3+C\rho$; respectively. Solving these Bernoulli equations we obtain the first integrals of statement (b).

Systems of statement (b) are Darbouxian integrable because their first integrals are obtained by integrating elementary functions, see for more details [19]. Concretely, these systems have a Darbouxian first integral taking into account that the integrals that appear in the first integrals of statement (b) can be computed using recurrent formulas, see for instance [16, p. 149]. □

Proof of Theorem 2(c). The proof follows easily taking $\lambda = 0$ in statements (a) and (b). $\square$

Proof of Theorem 2(d). If $f(\theta) \equiv 0$ then we have that system (3) with $\lambda = 0$ has $\dot{r} = 0$ and therefore it has the polynomial first integral $H = x^2 + y^2$. So statement (d.1) follows.

If $g(\theta) \equiv 0$, from the Abel differential equation (6) it is easy to obtain that $H(\rho, \theta) = 1/\rho + (n-1)\int f(\theta)\, d\theta$ is a first integral. Taking into account that $\int_0^{2\pi} f(\theta)\, d\theta = 0$ and going back to the cartesian variables, we obtain a rational first integral and statement (d.2) follows.

If $B(\theta) \equiv 0$, again from the Abel differential equation (6) it is easy to obtain that $H(\rho, \theta) = 1/\rho^2 - g(\theta)^2$ is a first integral. Going back to the cartesian variables we obtain a rational first integral, and statement (d.3) follows.

Finally, from the expression of the first integral $H(\rho, \theta)$ for $a < 1/4$ and $a \neq 0$ with $\sqrt{1-4a}$ rational, we have $H^2(\rho, \theta) = \rho^2 |\sqrt{1-4a} + F|^{2\mu_1} |\sqrt{1-4a} - F|^{2\mu_2}$, and a convenient power of it gives a rational first integral. So, it follows statement (d.4). $\square$

Now we study if it is possible to find other integrable cases from the well-known integrable cases of the Abel differential equation. Following the case (a) of Abel differential equation studied in [11, p. 24], first we do the change of variables $(\rho, \theta) \to (\eta, \xi)$ defined by $\rho = w(\theta)\eta(\xi) - B(\theta)/(3A(\theta))$, where $w(\theta) = \exp[\int [C - B^2(\theta)/(3A(\theta))]\, d\theta]$ and $\xi = \int A(\theta) w^2(\theta)\, d\theta$. This transformation writes the Abel equation into the normal form

$$\eta'(\xi) = [\eta(\xi)]^3 + I(\theta), \tag{13}$$

where

$$I(\theta) = \frac{1}{A(\theta) w^3(\theta)} \left[\frac{d}{d\theta} \left(\frac{B(\theta)}{3A(\theta)} \right) - \frac{CB(\theta)}{3A(\theta)} + \frac{2B^3(\theta)}{27A^2(\theta)} \right].$$

From the definition of $w(\theta)$ we have

$$\ln |w(\theta)| = \int \left[C - \frac{B(\theta)^2}{3A(\theta)} \right] d\theta = \int \frac{B(\theta)}{A(\theta)} \left[\frac{CA(\theta)}{B(\theta)} - \frac{B(\theta)}{3} \right] d\theta. \tag{14}$$

Since $a \neq 0$, using (7) or equivalently (12) in (14) and taking into account that $C = (n-1)\lambda$, we obtain

$$-\frac{1}{3a} \int \frac{\frac{d}{d\theta}\left(\frac{A(\theta)}{B(\theta)}\right)}{\frac{A(\theta)}{B(\theta)}} d\theta + \left(1 - \frac{1}{3a}\right) \int C\, d\theta = -\frac{1}{3a} \ln \left| \frac{A(\theta)}{B(\theta)} \right| + \left(1 - \frac{1}{3a}\right)(n-1)\lambda\theta.$$

And using this result $w(\theta) = |A(\theta)/B(\theta)|^{-1/3a} \exp((n-1)(1-1/(3a))\lambda\theta)$ and therefore $I(\theta)$ becomes

$$I(\theta) = \left[\frac{2-9a}{27}\right]\left(\frac{A(\theta)}{B(\theta)}\right)^{(1-3a)/a} \exp((n-1)(1-3a)\lambda\theta/a). \tag{15}$$

It is easy to see that for $a = 2/9$ and for $a = 1/3$ we have $I(\theta) = 0$ and $I(\theta) = -1/27$, respectively. For these two cases, the differential equation (13) is of separable variables and we can obtain the associated first integrals. But $I(\theta) = 0$ and $I(\theta) = -1/27$ implies that equality (7) holds with $a = 2/9$ and for $a = 1/3$, respectively. So we obtain cases already studied. New cases of integrability would be able to appear for $I(\theta) \neq 0, -1/27$.

We must mention that cases (b) and (c) of Abel differential equation studied in [11, p. 25] provide again the case studied for $a = 2/9$.

3. Algebraic limit cycles with Darbouxian first integral

In the next proposition we present probably the easiest example of a polynomial differential system which has a Darbouxian first integral and an algebraic limit cycle. Different examples of this kind were given by Dolov [8], Kooij and Christopher [12], and Christopher [6].

Proposition 4. *The differential system*

$$x' = x - y - x(x^2 + y^2), \quad y' = x + y - y(x^2 + y^2) \tag{16}$$

has the algebraic solution $x^2 + y^2 - 1 = 0$ as a limit cycle. In polar coordinates (2) *the function $H(r,\theta) = (r^2-1)\exp(2\theta)/r^2 = C$ is a Darbouxian first integral defined on $\mathbb{R}^2 \backslash \Sigma$ where $\Sigma = (\{(0,0)\} \cup \{(x,y)\colon x^2 + y^2 - 1 = 0\}$. We remark that H is not continuous on Σ.*

Proof. See [3]. $\square$

System (16) is of the class HN_f with $a = 0$.

In order to study the existence and non-existence of the limit cycles of system (1) we shall need the following result.

Theorem 5. *Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$. Let $V = V(x,y)$ be a C^1 solution of the linear partial differential equation*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V,$$

defined in U. If γ is a limit cycle of (P, Q) contained in U, *then* γ *is contained in* $\{(x, y) \in U\colon\ V(x, y) = 0\}$.

Proof. See Theorem 9 of [10]. $\square$

We note that under the assumptions of Theorem 5 the function $1/V$ is an integrating factor in $U \setminus \{V(x, y) = 0\}$. Again for more details, see [3,7]. So, the function V is called an *inverse integrating factor*.

Proof of Theorem 3. For systems (1) in the class HN_f with $\lambda \neq 0$ and $A(\theta)B(\theta) \neq 0$, it is easy to check that an inverse integrating factor of its associated Abel differential equation (6) is given by

$$V(\rho, \theta) = \rho(\rho^2 A(\theta)^2 / B(\theta)^2 + \rho A(\theta)/B(\theta) + a).$$

Notice that V is defined for all (ρ, θ) such that $B(\theta) \neq 0$. By Proposition 1 and Theorem 5, if system (1) and consequently its associated Abel equation (6) have limit cycles, those of the Abel equation must be contained into the set $\{V(\rho, \theta) = 0\}$.

From the expression of the inverse integrating factor, the unique possible limit cycles must be given by

$$\rho(\theta) = \begin{cases} \dfrac{(-1 \pm \sqrt{1-4a})B(\theta)}{2A(\theta)} & \text{if } a < \dfrac{1}{4}, \\[2ex] -\dfrac{B(\theta)}{2A(\theta)} & \text{if } a = \dfrac{1}{4}. \end{cases}$$

In order that these expressions of $\rho(\theta)$ define limit cycles, we must have that $\rho(\theta)$ is defined for all θ and that $\rho(\theta) > 0$ for all θ. So, since $A(\theta)$ and $B(\theta)$ are homogeneous trigonometric polynomials of degree $2(n+1)$ and $(n+1)$, respectively, we conclude that n must be odd and $a \leqslant 1/4$.

Clearly our class of systems (1) have no limit cycles if n is even, or if $a > 1/4$. They can have one limit cycle if $a = 1/4$, thus for instance system (j) for $k = 0$ in Corollary A.2 has exactly one limit cycle given by the circle $1 - 2(x^2 + y^2) = 0$. Moreover, Proposition 4 provides another example of a system of class HN_f having exactly one algebraic limit cycle.

In order to end the proof of Theorem 3(a) we must show that never the two expressions $\rho(\theta) = (-1 \pm \sqrt{1-4a})B(\theta)/(2A(\theta))$ provide two algebraic limit cycles. If we pass these two solutions of the Abel equation (6), to polar coordinates (r, θ) we get

$$r(\theta) = 2^{\frac{1}{-1+n}} \left(- \left(\frac{a\, B(\theta)}{A(\theta) \pm \sqrt{1-4a} A(\theta) + 2aB(\theta)g(\theta)} \right) \right)^{\frac{1}{-1+n}}.$$

The product of the two denominators of this expression is zero if and only if $\rho(\theta) = 1/g(\theta)$ is a solution of the Abel equation (4), see for more details the expression of the function $V(\rho, \theta)$. It is easy to check, by direct substitution, that $\rho(\theta) = 1/g(\theta)$ is a solution of the Abel equation (6). Therefore, the proof of Theorem 3(a) is completed. □

We note that the solution $\rho(\theta) = 1/g(\theta)$ of the Abel equation (6) corresponds to the infinity of system (1) in its Poincaré compactification.

Appendix

Following the work of Bautin [1], a quadratic system with a focus or a center may be written through an affine change of variables and a time rescaling into the form

$$\begin{aligned} \dot{x} &= \lambda_1 x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \end{aligned} \tag{A.1}$$

where λ_i are arbitrary constants for $i = 1, \dots, 6$. We work with this normal form for the quadratic systems because this form is one or the most used for studying the center problem for the quadratic systems, and using it the conditions for a center take a very simple form.

Corollary A.1. *Quadratic system* (A.1) *with* $\lambda_1 \neq 0$ *belong to the class* HN_f *if one of the following statements holds*

(a) $\lambda_2 = \lambda_5 = 0,\ \lambda_4 = -4\lambda_3,\ \lambda_6 = \lambda_3,\ a = (1 + \lambda_1^2)/(4\lambda_1^2)$. *Then,* (A.1) *has the Darbouxian first integral*

$$H(x, y) = \frac{(x^2 + y^2)\exp[-2\lambda_1\ \arctan[\frac{\lambda_1 x + y - \lambda_3(x^2+y^2)}{x - \lambda_1 y}]]}{1 + \lambda_1^2 - 2\lambda_1\lambda_3 x - 2\lambda_3 y + \lambda_3^2(x^2 + y^2)}.$$

(b) $\lambda_2 = \lambda_5 = 0,\ \lambda_4 = -2\lambda_3,\ \lambda_6 = -\lambda_3,\ a = 1/4$. *Then,* (A.1) *has the Darbouxian first integral*

$$H(x, y) = (x^2 + y^2)\exp\left[2\left(1 + \lambda_3 y - \lambda_1\ \arctan\left[\frac{y}{x}\right]\right)\right].$$

(c) $\lambda_2 = \lambda_5 = \lambda_6 = 0,\ \lambda_4 = -3\lambda_3,\ a = 0$. *Then,* (A.1) *has the Darbouxian first integral*

$$H(x, y) = \frac{(x^2 + y^2)\exp[-2\lambda_1\ \arctan[\frac{y}{x}]]}{(-1 - \lambda_1^2 + \lambda_1\lambda_3 x + \lambda_3 y)^2}.$$

(d) $\lambda_3 = -\lambda_1\lambda_2,\ \lambda_4 = 2\lambda_1\lambda_2,\ \lambda_5 = -4\lambda_2,\ \lambda_6 = \lambda_1\lambda_2,\ a = (1+\lambda_1^2)(\lambda_1^2-1)/(4\lambda_1)$. *Then*, (A.1) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)\exp[2\lambda_1\ \arctan[\frac{x}{y}]]}{(1+\lambda_1^2-2\lambda_2 x-2\lambda_1\lambda_2 y)^{1-\lambda_1^2}}.$$

(e) $\lambda_3 = -\lambda_1\lambda_2/3,\ \lambda_4 = 4\lambda_1\lambda_2/3,\ \lambda_5 = -4\lambda_2,\ \lambda_6 = -\lambda_1\lambda_2/3,\ a = (1+\lambda_1^2)/(4\lambda_1^2)$. *Then*, (A.1) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)\exp[-2\lambda_1\ \arctan[\frac{3\lambda_1 x+3y+\lambda_1\lambda_2(x^2+y^2)}{3(x-\lambda_1 y+\lambda_2(x^2+y^2))}]]}{9+9\lambda_1^2+6\lambda_2(3+\lambda_1^2)x-12\lambda_1\lambda_2 y+\lambda_2^2(9+\lambda_1^2)(x^2+y^2)}.$$

(f) $\lambda_3 = -(3+\lambda_1^2)\lambda_2/(2\lambda_1),\ \lambda_4 = 3(1+\lambda_1^2)\lambda_2/(2\lambda_1),\ \lambda_5 = -4\lambda_2,\ \lambda_6 = 3\lambda_2/\lambda_1,\ a = 2/9$. *Then*, (A.1) *has the Darbouxian first integral*

$$H(x,y) = (x^2+y^2)\exp\left[-2\lambda_1\ \arctan\left[\frac{y}{x}\right]\right](2\lambda_1+\lambda_1\lambda_2 x-3\lambda_2 y)^2.$$

(g) $\lambda_3 = -\lambda_1\lambda_2/2,\ \lambda_4 = 2\lambda_1\lambda_2,\ \lambda_5 = -4\lambda_2,\ \lambda_6 = -\lambda_1\lambda_2/2,\qquad a = (1+\lambda_1^2)/(4\lambda_1^2)$. *Then*, (A.1) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)\exp[-2\lambda_1\ \arctan[\frac{2\lambda_1 x+2y+\lambda_1\lambda_2(x^2+y^2)}{2(x-\lambda_1 y+\lambda_2(x^2+y^2))}]]}{4+4\lambda_1^2+4\lambda_2(2+\lambda_1^2)x-4\lambda_1\lambda_2 y+\lambda_2^2(4+\lambda_1^2)(x^2+y^2)}.$$

(h) $\lambda_3 = -2(1+\lambda_1^2)\lambda_2/(3\lambda_1),\ \lambda_4 = 2\lambda_1\lambda_2,\ \lambda_5 = -4\lambda_2, \lambda_6 = 2\lambda_2/\lambda_1,\ a = 3/16$. *Then*, (A.1) *has the Darbouxian first integral*

$$H(x,y) = (x^2+y^2)\exp\left[-2\lambda_1\ \arctan\left[\frac{y}{x}\right]\right](3\lambda_1+2\lambda_1\lambda_2 x-4\lambda_2 y).$$

Consequently, *for* $\lambda_1 \neq 0$ *these quadratic systems have a focus at the origin and are Darbouxian integrable*.

The proof of Corollary A.1 follows doing tedious computations using statements (a) and (b) of Theorem 2 when $n = 2$.

Corollary A.2. *System* (8) *with* $\lambda_1 \neq 0$ *belong to the class* HN_f *if one of the following statements holds*

(a) $\lambda_2 = \lambda_3 = \lambda_5 = \lambda_6 = \lambda_7 = 0,\ \lambda_4 = \lambda_8,\ a = 0$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)\exp[-2\lambda_1\ \arctan[\frac{y}{x}]]}{2\lambda_1+2\lambda_1^3+\lambda_8 x^2-2\lambda_1\lambda_8 xy+\lambda_8(1+2\lambda_1^2)y^2}.$$

(b) $\lambda_2 = -3\lambda_3/(4\lambda_1)$, $\lambda_4 = -(9\lambda_3 + 16\lambda_1^2\lambda_3)/(4\lambda_1)$, $\lambda_5 = 2\lambda_3$, $\lambda_6 = 0$, $\lambda_7 = 9\lambda_3/(4\lambda_1)$, $\lambda_8 = -(3\lambda_3 + 16\lambda_1^2\lambda_3)/(4\lambda_1)$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)^2 \exp[-4\lambda_1 \ \arctan[\frac{y}{x}]]}{(\lambda_1 + 3\lambda_3 xy - 4\lambda_1\lambda_3^2 y^2)}.$$

(c) $\lambda_2 = -\lambda_8$, $\lambda_3 = 0$, $\lambda_4 = 3\lambda_8$, $\lambda_5 = \lambda_6 = 0$, $\lambda_7 = -3\lambda_8$, $a = (1+\lambda_1^2)/(4\lambda_1^2)$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)^2 \exp[2\lambda_1 \ \arctan[\frac{-\lambda_1 x^2 - 2xy + \lambda_1 y^2 + \lambda_8 (x^2+y^2)^2}{x^2 - 2\lambda_1 xy - y^2}]]}{1 + \lambda_1^2 - 2\lambda_1\lambda_8 x^2 - 4\lambda_8 xy + 2\lambda_1\lambda_8 y^2 + \lambda_8 (x^2+y^2)^2}.$$

(d) $\lambda_3 = -\lambda_1\lambda_2/3$, $\lambda_4 = -2\lambda_1^2\lambda_2/3$, $\lambda_5 = -2\lambda_1\lambda_2/3$, $\lambda_6 = \lambda_7 = 0$, $\lambda_8 = -\lambda_2(3+2\lambda_1^2)/3$, $a = 2/9$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)^2 \exp[-2\lambda_1 \ \arctan[\frac{y}{x}]]}{(3 + 3\lambda_2 xy - 2\lambda_1\lambda_2 y^2)^2}.$$

(e) $\lambda_2 = 3\lambda_7$, $\lambda_3 = -4\lambda_1\lambda_7/5$, $\lambda_4 = -(5+16\lambda_1^2)\lambda_7/5$, $\lambda_5 = -8\lambda_1\lambda_7/5$, $\lambda_6 = 0$, $\lambda_8 = -(15+16\lambda_1^2)\lambda_7/5$, $a = 3/16$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2)^2 \exp[-4\lambda_1 \ \arctan[\frac{y}{x}]]}{(5 + 4\lambda_7 y(5x - 4\lambda_1 y))^3}.$$

(f) $\lambda_2 = \lambda_7$, $\lambda_3 = 0$, $\lambda_4 = \lambda_8$, $\lambda_5 = \lambda_6 = 0$, $a = 0$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2) \exp[-2\lambda_1 \ \arctan[\frac{y}{x}]]}{2\lambda_1 + 2\lambda_1^3 + 2\lambda_1(\lambda_7 - \lambda_8)xy + (\lambda_7+\lambda_8)(x^2+y^2) + 2\lambda_1^2(\lambda_7 x^2 + \lambda_8 y^2)}.$$

(g) $\lambda_3 = -\lambda_1(\lambda_2^2 - \lambda_7^2)/(3\lambda_2 + \lambda_7)$, $\lambda_4 = -(2\lambda_1^2(\lambda_2 + 2\lambda_7)^2 + \lambda_7(3\lambda_2+\lambda_7))/(3\lambda_2+\lambda_7)$, $\lambda_5 = -2\lambda_1(\lambda_2^2 - \lambda_7^2)/(3\lambda_2+\lambda_7)$, $\lambda_6 = 0$, $\lambda_8 = -(\lambda_2^2(3+2\lambda_1^2) + 2\lambda_1^2\lambda_7^2 + \lambda_2\lambda_7(1+4\lambda_1^2))/(3\lambda_2+\lambda_7)$, $a = 2\lambda_2(\lambda_2 - \lambda_7)/(3\lambda_2 - \lambda_7)^2$. *Then*, (8) *has the Darbouxian first integral*

$$H(x,y) = \frac{(x^2+y^2) \exp[-2\lambda_1 \ \arctan[\frac{y}{x}]]}{[(3\lambda_2+\lambda_7)(1 + (\lambda_2+\lambda_7)xy) - 2\lambda_1(\lambda_2+\lambda_7)^2 y^2]^{\frac{2\lambda_2}{\lambda_2+\lambda_7}}}.$$

(h) $\lambda_2 = \lambda_1\lambda_6$, $\lambda_3 = -(3+4\lambda_1^2)\lambda_6/3$, $\lambda_4 = \lambda_1(15+16\lambda_1^2)\lambda_6/3$, $\lambda_5 = -(3+8\lambda_1^2)\lambda_6/3$, $\lambda_7 = -\lambda_1\lambda_6$, $\lambda_8 = \lambda_1(9+16\lambda_1^2)\lambda_6/3$. *Then*, (8) *has the Darbouxian first* *integral*

$$H(x,y)=\frac{(x^2+y^2)^2\exp[-4\lambda_1\ \arctan[\frac{y}{x}]]}{3+2\lambda_6(3x^2-6\lambda_1xy+(3+8\lambda_1^2)y^2)}.$$

(i) $\lambda_2=\lambda_7=\lambda_1\lambda_6/2,\ \lambda_4=\lambda_8=-\lambda_1(2\lambda_3+\lambda_6)/2,\quad \lambda_5=2\lambda_3+\lambda_6,\ a=2/9.$ *Then*, (8) *has the Darbouxian first integral*

$$H(x,y)=\frac{(x^2+y^2)\exp[-2\lambda_1\ \arctan[\frac{\lambda_1x+y-\lambda_3(x^2+y^2)}{x-\lambda_1y}]]}{1+\lambda_1^2-2\lambda_1\lambda_3x-2\lambda_3y+\lambda_3^2(x^2+y^2)}.$$

(j) $\lambda_2=\lambda_4=\lambda_7=\lambda_8=\lambda_1\lambda_5(k\lambda_5-2)/(k\lambda_5-1),\ \lambda_3=\lambda_5,\ \lambda_6=-\lambda_5,\ a=(1-k\lambda_5)/\lambda_5^2.$ *Then*, (8) *has the Darbouxian first integral*

$$H(x,y)=\frac{(x^2+y^2)\exp[-2\lambda_1\ \arctan[\frac{y}{x}]]}{(1-k\lambda_5+\lambda_5(k\lambda_5-2)(x^2+y^2))^{\frac{1}{k\lambda_5-2}}}.$$

Consequently, *for* $\lambda_1\neq 0$ *these cubic systems have a focus at the origin and are Darbouxian integrable.*

The proof of Corollary A.2 follows doing tedious computations using statements (a) and (b) of Theorem 2 when $n=3$.

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